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Polarisation of the vacuum near a black hole inside a spherical cavity

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Abstract. The renormalised expectation value in the Hartle–Hawking vacuum of the stress–energy tensor of a massless conformally coupled scalar field is discussed near the horizon of a Schwarzschild black hole inside a spherical cavity. The values of radial pressure, tangential pressure and energy density depend on the coordinate radius of the cavity. For large cavities we recover the results of Candelas, i.e. we obtain a negative energy density near the horizon. As the cavity radius decreases, the energy density first decreases to a minimum and then increases through zero to positive values.

1. Introduction

The problem of calculating the expectation value of the stress–energy tensor of a quantum field in a curved space–time can be solved by using standard regularisation methods such as point-splitting procedures (Christensen 1976, 1978, Adler *et al* 1977, Adler and Lieberman 1978, Wald 1977, 1978). In many cases there are, however, technical difficulties associated with the evaluation of the expectation value. For instance, the mode functions corresponding to the various proposed vacuum states of a quantum field propagating in a black hole space–time (Unruh 1976, Israel 1976, Gibbons and Perry 1978, Fulling 1977, Iyer and Kumar 1979a, 1979b, Sciama *et al* 1981) cannot be completely expressed in terms of known functions. Since the point-separated expectation value is a distribution, numerical methods can hardly be applied in general. Therefore, up to now few attempts have been made to obtain the stress–energy tensor of a quantum field interacting with a black hole (Christensen and Fulling 1977, Candelas 1980, Page 1982, Fawcett and Whiting 1982, Elster 1982a) the complete form of which is necessary for, e.g., solving the back-reaction problem of black hole evaporation (Candelas 1980, Kodama 1980, Hajicek and Israel 1980, Bardeen 1981, Frolov 1981, Hiscock 1981).

Fortunately, in performing renormalisation calculations in black hole metrics one can take advantage of the fact that in these metrics all (integer spin) fields which are analytic solutions of the corresponding field equations in the whole manifold turn out to be periodic in imaginary time with period $2\pi/\kappa$, where κ is the surface gravity of the black hole (Hartle and Hawking 1976, Gibbons and Perry 1976, 1978, Hawking 1981). This has been used by a number of authors (Candelas 1980, Frolov 1982, Elster 1982b) to examine the vacuum polarisation near the horizon of a black hole.

Since a black hole emits particles like a hot body of temperature $\kappa/2\pi$, it can be in thermal equilibrium with a gas of massless quanta. This equilibrium is, however, only stable if the whole system is placed in a sufficiently small box with perfectly reflecting walls (Hawking 1976, Gibbons and Perry 1978, Wilkins 1979). The presence of the walls not only stabilises the thermal equilibrium between black hole and radiation field, but also modifies the expectation value of the stress–energy tensor of the quantum field in question. For a box which is large compared with the size of the black hole, the boundary correction to the stress–energy tensor can be shown to be important only in a region near the boundary (Elster 1982a). However, for small boxes the boundary may alter the expectation values of energy density and pressure even in the vicinity of the horizon. As is well known (Dowker and Kennedy 1978, Deutsch and Candelas 1979, Kennedy *et al* 1980), near a boundary the renormalised stress–energy tensor of a conformally coupled scalar field varies with the inverse third power of the distance from the boundary, provided that Dirichlet boundary conditions are imposed.

The aim of this paper is to discuss the vacuum polarisation near the horizon of a spherically symmetric black hole when the hole is at the centre of a spherical cavity. We therefore generalise some results of a paper by Candelas (1980) which are recovered for large radii of the cavity. This generalisation is desirable for at least two reasons. As already mentioned, the presence of the boundary is necessary to stabilise the thermal equilibrium between black hole and quantum field. Moreover, any situation in which the presence of a boundary contributes to the renormalised expectation values of physical quantities in a curved space–time is of particular interest.

Throughout we closely follow Candelas (1980) in his approach to the renormalisation problem. Several technical details from his paper are used without mentioning this in any case. We confine ourselves to a scalar massless conformally coupled field ϕ.

2. Renormalisation of the stress–energy tensor

Let $p_r(r) = \langle 0|T^r_r|0\rangle_{\text{ren}}$, $p_t(r) = \langle 0|T^\vartheta_\vartheta|0\rangle_{\text{ren}} = \langle 0|T^\varphi_\varphi|0\rangle_{\text{ren}}$ and $\mu(r) = -\langle 0|T^t_t|0\rangle_{\text{ren}}$ be the renormalised values of radial pressure, tangential pressure and energy density in the Hartle–Hawking vacuum (r, ϑ, φ and t denote the usual Schwarzschild coordinates). Since all physical quantities are well behaved in this vacuum on both the past and future horizons (with respect to a physically reasonable frame), the quantities p_r, p_t and μ remain finite as $r \to 2M$ and must satisfy

$$p_r + \mu \xrightarrow[r \to 2M]{} 0, \tag{2.1}$$

where M is the mass of the black hole. (Units with $G = c = \hbar = k = 1$ are used.) We employ the DeWitt–Christensen technique of point-splitting regularisation (Christensen 1976, 1978). Separating points along radial geodesics, we may write the renormalised quantities in the form

$$p_r(r) = \lim_{r' \to r} [\tilde{p}_r(r, r') + p_r^{\text{inf}}(r, r')] + p_r^{\text{fin}}(r) \tag{2.2}$$

(similarly for p_t and μ), where quantities with tildes are the point-separated expectation values, and the superscripts inf and fin are used to denote the divergent and the finite (direction-dependent) counterterms, respectively. From Christensen's (1976) paper we conclude that the counterterms should take the form

$$p_r^{\text{inf}} = -3p_t^{\text{inf}} = 3\mu^{\text{inf}} = (3/2\pi^2)\varepsilon^{-4}(r, r') \tag{2.3a}$$

$$p_r^{\text{fin}} = (1/48\pi^2)(M^2/r^6), \qquad p_t^{\text{fin}} = (11/360\pi^2)(M^2/r^6) - (1/60\pi^2)(M/r^5),$$
$$\mu^{\text{fin}} = (47/720\pi^2)(M^2/r^6) - (1/30\pi^2)(M/r^5), \tag{2.3b}$$

where $\varepsilon(r, r')$ is the geodesic distance between two radially separated points with coordinates r and r'. Equation (2.3*b*) leads to the trace anomaly of the stress–energy tensor of a massless conformally coupled scalar field in the Schwarzschild space–time,

$$p_r + 2p_t - \mu = (1/60\pi^2)(M^2/r^6). \tag{2.4}$$

Because of (2.1) and (2.4), near the horizon the number of unknowns which are to be determined is reduced to one. For later purposes the following expansion of (2.3*a*) proves to be useful:

$$\varepsilon^{-4}(r, r' = 2M) = (1/64M^2)(r-2M)^{-2}$$
$$- (1/192M^3)(r-2M)^{-1} + 17/11\,520M^4 + \mathrm{O}(r-2M). \tag{2.5}$$

3. Pressure and energy density near the horizon

Finite-temperature quantum field theory in a Schwarzschild space–time can be conveniently studied by performing a Wick rotation $t = -\mathrm{i}\xi$ to imaginary values of the time coordinate. In the coordinates $(r, \vartheta, \varphi, \xi)$ the Hartle–Hawking propagator $G_{\mathrm{H}}(r, \vartheta, \varphi, -\mathrm{i}\xi; r', \vartheta', \varphi', -\mathrm{i}\xi')$ is a solution of

$$G_{\mathrm{H};\alpha}{}^{;\alpha} = -\mathrm{i}r^{-2}\sin^{-1}\vartheta\,\delta(r-r')\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')\delta(\xi-\xi') \tag{3.1}$$

which is periodic in ξ with period $8\pi M$. The point-separated expectation value of the stress–energy tensor results from the solution of (3.1) in the usual way:

$$\langle 0|\tilde{T}_{\alpha\beta}|0\rangle = -(\mathrm{i}/3)(G_{\mathrm{H},\alpha,\beta'}g^{\beta'}{}_{\beta} + G_{\mathrm{H},\alpha',\beta}g^{\alpha'}{}_{\alpha})$$
$$+ (\mathrm{i}/6)(G_{\mathrm{H},\gamma,\gamma'}g^{\gamma\gamma'}g_{\alpha\beta} + G_{\mathrm{H},\alpha;\beta} + G_{\mathrm{H},\alpha';\beta'}g^{\alpha'}{}_{\alpha}g^{\beta'}{}_{\beta}). \tag{3.2}$$

(In the last equation the metric is assumed to satisfy Einstein's vacuum field equations.) On the Euclidean section of the analytically continued space–time the only non-vanishing components of the parallel propagator relating quantities on radial geodesics are

$$g_{rr'} = (rr')^{1/2}(r-2M)^{-1/2}(r'-2M)^{-1/2}, \qquad g_{\vartheta\vartheta'} = rr',$$
$$g_{\varphi\varphi'} = rr'\sin^2\vartheta, \qquad g_{\xi\xi'} = (r'-2M)^{1/2}(r'-2M)^{1/2}(rr')^{-1/2}. \tag{3.3}$$

As shown by Candelas (1980), the solution of (3.1) may be written as a Fourier series the coefficients of which take the form of multipole expansions,

$$G_{\mathrm{H}} = (\mathrm{i}/16\pi^2 Mr)\sum_{l=-\infty}^{\infty}\exp[\mathrm{i}l(\xi-\xi')/4M]\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{1/2}P_n(\cos\gamma)R_n^l(r, r'),$$
$$\cos\gamma = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi-\varphi'). \tag{3.4}$$

The radial functions R_n^l satisfy

$$\{r(r-2M)\,\mathrm{d}^2/\mathrm{d}r^2 + 2(r-M)\,\mathrm{d}/\mathrm{d}r - n(n+1)$$
$$- l^2r^3/[16M^2(r-2M)]\}r^{-1}R_n^l(r, r') = -\delta(r-r'). \tag{3.5}$$

Let r_0 be the coordinate radius of the cavity. Imposing Dirichlet boundary conditions on the field, we are interested in solutions of (3.5) which vanish at $r = r_0$. It is convenient to split the radial functions into a part ${}_1R^l_n$ (which is an appropriate solution of (3.5) if the boundary is absent) and a boundary correction ${}_2R^l_n$. The contribution of ${}_1R^0_n$ to G_H corresponds to the field of a scalar point charge located at the point $(r', \vartheta', \varphi')$ outside the black hole. The boundary correction ${}_2R^0_n$ is a solution of the homogeneous equation corresponding to (3.5), i.e. it corresponds to a field of scalar charges located in the region $r > r_0$. Using the variable $\eta = r/M - 1$, we obtain in the case $l = 0$ (cf Candelas 1980, Elster 1982b)

$$r^{-1}{}_1R^0_n(r, r') = M^{-1}(n+\tfrac{1}{2})^{-1/2}\begin{cases} Q_n(\eta')P_n(\eta), & r < r', \\ P_n(\eta')Q_n(\eta), & r > r', \end{cases} \tag{3.6a}$$

$$r^{-1}{}_2R^0_n(r, r') = -M^{-1}(n+\tfrac{1}{2})^{-1/2}P_n(\eta')P_n(\eta)Q_n(\eta_0)/P_n(\eta_0), \tag{3.6b}$$

where the P_n and Q_n are Legendre functions of the first and second kind, respectively. For $|l| \geq 1$ we have

$$r^{-1}{}_1R^l_n(r, r') = (2|l|M)^{-1}(n+\tfrac{1}{2})^{-1}\begin{cases} q^l_n(\eta')p^l_n(\eta), & r < r', \\ p^l_n(\eta')q^l_n(\eta), & r > r', \end{cases} \tag{3.7a}$$

$$r^{-1}{}_2R^l_n(r, r') = -(2|l|M)^{-1}(n+\tfrac{1}{2})^{-1}p^l_n(\eta')p^l_n(\eta)q^l_n(\eta_0)/p^l_n(\eta_0). \tag{3.7b}$$

The p^l_n and q^l_n cannot be expressed in terms of known functions. (The functions P_n, Q_n, p^l_n and q^l_n are defined as in Candelas (1980) up to normalisation; some of their properties are given in the appendix.)

Using equations (3.3), (3.4), (3.6a, b), (3.7a, b), (A1) and (A6), we are now able to calculate the limit $r' \to 2M$ in equation (3.2). Since each point $(r' = 2M, \vartheta', \varphi', \xi')$ is a fixed point of the symmetry group generated by the Killing vector $\partial/\partial\xi'$, only terms with $l = 0, 1, 2$ contribute to the point-separated expectation value in this limit. Because of (2.1) and (2.4) it suffices to evaluate one component of the stress–energy tensor. For convenience, the tangential pressure $p_t = {}_1p_t + {}_2p_t$ is calculated because it involves no $l = 2$ term. If the boundary were absent, we would obtain from (3.2) in the partial coincidence limit $\vartheta' \to \vartheta$, $\varphi' \to \varphi$, $\xi' \to \xi$

$$\begin{aligned} {}_1\tilde{p}_t(r, r' = 2M) &= -(768\pi^2M^4r^2)^{-1}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})[8M(r-2M)\,\mathrm{d}Q_n/\mathrm{d}\eta \\ &\quad -(r+2M)^2n(n+1)Q_n(\eta) \\ &\quad +2\sqrt{2}r^{3/2}(r-2M)^{1/2}(n+\tfrac{1}{2})^{-1/2}\,\mathrm{d}q^1_n/\mathrm{d}\eta \\ &\quad +(1/\sqrt{2})r^{5/2}(r-2M)^{-1/2}(n+\tfrac{1}{2})^{-1/2}q^1_n(\eta)]. \end{aligned} \tag{3.8a}$$

The boundary correction to the tangential pressure requires no renormalisation, thus we can also perform the limit $r \to 2M$ to obtain

$$\begin{aligned} {}_2p_t(2M) &= (384\pi^2M^4)^{-1}\sum_{n=0}^{\infty}[(n+\tfrac{1}{2})q^1_n(\eta_0)/n^1_n(\eta_0) \\ &\quad -2n(n+1)(n+\tfrac{1}{2})^{3/2}Q_n(\eta_0)/P_n(\eta_0)]. \end{aligned} \tag{3.8b}$$

Since the $q^1_n(\eta)$ asymptotically approach the Legendre functions $Q^1_n(\eta)$ as $\eta \to 1$ (see appendix), it is now useful to set $q^1_n = Q^1_n + \hat{q}^1_n$ in (3.8a). The advantage of this splitting is that the summation over n can be carried out in all terms on the right-hand side of (3.8a) which are singular as $r \to 2M$. (This can be accomplished by using (A5a, b) and (A14a).) Taking into account the counterterms (2.3a) and (2.3b) and the expansion (2.5), we arrive at the renormalised expectation value

$$ {}_1p_t(2M) = -(768\pi^2M^4)^{-1}\Big(\tfrac{1}{10} - \lim_{r\to 2M}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{1/2}[-2\sqrt{2}(r-2M)^{1/2}r^{-1/2}\,\mathrm{d}\hat{q}_n^1/\mathrm{d}\eta - (1/\sqrt{2})r^{1/2}(r-2M)^{-1/2}\hat{q}_n^1(\eta)]\Big). \tag{3.9} $$

In order to perform the limit in (3.9) we need an asymptotic expansion of the $\hat{q}_n^1$ as $\eta \to 1$. Equations (A11) and (A13) yield

$$ \hat{q}_n^1(\eta) = (n+\tfrac{1}{2})^{1/2}\{2I_n - \tfrac{1}{2}n(n+1)[2\psi(n+1) + 2\gamma - \tfrac{3}{2}] + \tfrac{1}{4} - \tfrac{1}{2}\ln 2\}(\eta-1)^{1/2} + \mathrm{O}[(\eta-1)^{3/2}] + \tfrac{1}{2}(n+\tfrac{1}{2})^{1/2}(\eta-1)^{1/2}\ln(\eta-1)[1+\mathrm{O}(\eta-1)], \tag{3.10} $$

where $\psi(x) = (\mathrm{d}/\mathrm{d}x)\ln\Gamma(x)$, γ is Euler's constant, and the I_n are the integrals defined by (A12). Somewhat surprisingly, there is also a divergent term in (3.10) as $\eta \to 1$, which is proportional to $(\eta-1)^{1/2}\ln(\eta-1)$. Therefore, the limit $\eta \to 1$ cannot be performed in (3.9) by simply substituting the expansion (3.10) for $\hat{q}_n^1$. To circumvent this difficulty we remember equation (A2) of the appendix, which leads to the ansatz

$$ \hat{q}_n^1(\eta) = -(n+\tfrac{1}{2})^{1/2}(\eta-1)^{1/2}Q_n(\eta) + \hat{\hat{q}}_n^1(\eta), \tag{3.11} $$

the asymptotic expansion as $\eta \to 1$ of the $\hat{\hat{q}}_n^1$ being obtainable from (3.10) and (A2). Using (A5a) and (A5b) we now succeed in deriving a final expression for the total tangential pressure $p_t(2M) = {}_1p_t(2M) + {}_2p_t(2M)$ at the horizon. A straightforward calculation yields

$$ p_t(2M) = (768\pi^2M^4)^{-1}\Big(\tfrac{3}{20} - 2\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\{2I_n - [n(n+1)+1][\psi(n+1)+\gamma] + 3n(n+1)/4 + \tfrac{1}{4} - q_n^1(\eta_0)/p_n^1(\eta_0) + 2(n+\tfrac{1}{2})^{1/2}n(n+1)Q_n(\eta_0)/P_n(\eta_0)\}\Big). \tag{3.12} $$

This expression must be evaluated numerically in terms of the coordinate radius r_0 of the cavity. Following Candelas (1980) we calculate the functions $p_n^1(x)$ in the range $1 \leqslant x \leqslant 2$ by summing the series (A8), (A9). In the range $2 < x < \infty$ these functions are calculated by using a fourth-order Runge–Kutta procedure starting at $x = 2$ with the values obtained by summing the series (A8), (A9) and the corresponding series for $\mathrm{d}p_n^1/\mathrm{d}x$. The ratio $q_n^1(x)/p_n^1(x)$ is easily computed with the help of equation (A10). The values of the Legendre functions $P_n(x)$ and $Q_n(x)$ can be determined by using the integral representations (A3a) and (A3b).

Figure 1 shows radial pressure, tangential pressure and energy density at the horizon. The sum of $p_r(2M)$ and $p_t(2M)$ is independent of r_0. It is equal to half the anomalous trace, cf equation (2.4). For large r_0 the renormalised expectation value of the stress–energy tensor tends to the values $p_r(2M) = -\mu(2M) \approx 10.37 \times 10^{-6}M^{-4}$ and $p_t(2M) \approx 2.82 \times 10^{-6}M^{-4}$ which have been obtained by Candelas. The radial pressure has a maximum of $13.07 \times 10^{-6}M^{-4}$ at $r_0 = 4.07M$, whereas the tangential pressure has a minimum of $0.13 \times 10^{-6}M^{-4}$ there. Both pressures are equal for $r_0 = 2.92M$. Radial pressure and energy density have a common zero at $r_0 = 2.75M$.

It can easily be shown that $p_r(2M)$, $p_t(2M)$ and $\mu(2M)$ completely determine the renormalised stress–energy tensor in Kruskal coordinates at the point of intersection

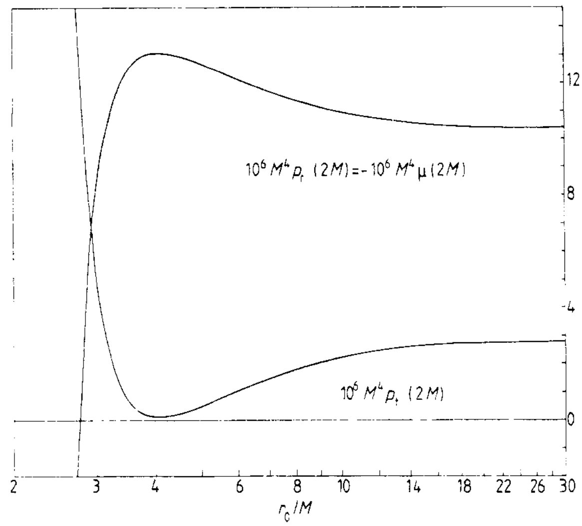

Figure 1. Radial pressure p_r, tangential pressure p_t and energy density μ at the horizon.

of the past and future horizons. However, at the rest of the horizon this tensor involves the derivative of $p_r+\mu$ at $r=2M$. Thus a further investigation providing the value of this derivative is necessary. (Using a suitable approximation, Page (1982) estimated this derivative in the limit $r_0 \to \infty$.)

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Appendix

In this appendix we give some properties of the radial functions, which are solutions of the homogeneous equation corresponding to (3.5). For $l=0$ the radial equation is solved by Legendre functions of the first and second kind, respectively. The behaviour of these functions in the neighbourhood of $x=1$ is

$$P_n(x) = (n+\tfrac{1}{2})^{1/2}[1+\tfrac{1}{2}n(n+1)(x-1)+\tfrac{1}{16}n(n+1)(n^2+n-2)(x-1)^2+\ldots], \tag{A1}$$

$$Q_n(x) = -\tfrac{1}{2}\ln(x-1)+\tfrac{1}{2}\ln 2-\psi(n+1)-\gamma+\ldots. \tag{A2}$$

For numerical computations one can take advantage of the integral representations (Erdélyi *et al* 1953)

$$P_n(x) = (n+\tfrac{1}{2})^{1/2}\pi^{-1}\int_0^\pi [x+(x^2-1)^{1/2}\cos t]^n \, dt, \tag{A3a}$$

$$Q_n(x) = 2^{-n-1}\int_0^\pi (x+\cos t)^{-n-1}(\sin t)^{2n+1} \, dt. \tag{A3b}$$

The functions P_n and Q_n obey Heine's formula

$$\tfrac{1}{2}(x-y)^{-1}=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{1/2}P_n(y)Q_n(x), \tag{A4}$$

from which we obtain

$$\tfrac{1}{2}(x-1)^{-1}=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})Q_n(x), \tag{A5a}$$

$$(x-1)^{-2}=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})n(n+1)Q_n(x). \tag{A5b}$$

For $|l|\geqslant 1$ the radial equation cannot be solved by known functions. Following Candelas (1980), we define two linearly independent solutions $p_n^l(x)$ and $q_n^l(x)$ by their behaviour in the neighbourhood of $x=1$ as follows:

$$p_n^l(x)=(n+\tfrac{1}{2})^{1/2}(x-1)^{|l|/2}\{1+(l^2-|l|+2n(n+1))/[4(|l|+1)](x-1)+\ldots\}, \tag{A6}$$

$$q_n^l(x)=(n+\tfrac{1}{2})^{1/2}(x-1)^{-|l|/2}+\ldots. \tag{A7}$$

In this paper we especially make use of the functions $p_n^1(x)$ and $q_n^1(x)$. In the range $1\leqslant x<3$ the $p_n^1(x)$ may be represented by the series

$$p_n^1(x)=(x-1)^{1/2}\sum_{k=0}^{\infty}c_k(x-1)^k, \tag{A8}$$

where the coefficients are given by

$$\begin{aligned}
&c_0=(n+\tfrac{1}{2})^{1/2}, \qquad c_1=(n+\tfrac{1}{2})^{1/2}n(n+1)/4,\\
&c_2=(n+\tfrac{1}{2})^{1/2}[2n^2(n+1)^2-6n(n+1)+3]/96,\\
&c_k=[2k(k+1)]^{-1}[(n^2+n-k^2+1)c_{k-1}+\tfrac{3}{8}c_{k-2}+\tfrac{1}{16}c_{k-3}], \qquad k\geqslant 3.
\end{aligned} \tag{A9}$$

From the Wronskian relation between the functions p_n^1 and q_n^1 it follows that the q_n^1 may be expressed in terms of the p_n^1 as follows:

$$q_n^1(x)=2(n+\tfrac{1}{2})p_n^1(x)\int_x^{\infty}[p_n^1(y)]^{-2}(y^2-1)^{-1}\,dy. \tag{A10}$$

From (A6) and (A10) we obtain an asymptotic expansion of the $q_n^1(x)$ as $x\to 1$, namely

$$\begin{aligned}
q_n^1(x)=&(n+\tfrac{1}{2})^{1/2}(x-1)^{-1/2}+(n+\tfrac{1}{2})^{1/2}[n(n+1)/4+2I_n](x-1)^{1/2}+O[(x-1)^{3/2}]\\
&-\tfrac{1}{2}(n+\tfrac{1}{2})^{1/2}[n(n+1)+1](x-1)^{1/2}\ln\frac{x+1}{x-1}\\
&\times\{1+n(n+1)(x-1)/4+O[(x-1)^2]\},
\end{aligned} \tag{A11}$$

where the abbreviation

$$I_n=\int_1^{\infty}\{(n+\tfrac{1}{2})[p_n^1(x)]^{-2}-(x-1)^{-1}+n(n+1)/2\}(x^2-1)^{-1}\,dx \tag{A12}$$

has been used.

For $\eta \approx 1$ the functions p_n^l and q_n^l may be approximated by Legendre functions P_n^l and Q_n^l. As $x \to 1$, the $Q_n^1(x)$ exhibit the asymptotic behaviour

$$Q_n^1(x) = (n+\tfrac{1}{2})^{1/2}(x-1)^{-1/2} + (n+\tfrac{1}{2})^{1/2}\{n(n+1)[\psi(n+1)+\gamma-\tfrac{1}{2}]-\tfrac{1}{4}\}(x-1)^{1/2}$$
$$+\mathrm{O}[(x-1)^{3/2}] - \tfrac{1}{2}(n+\tfrac{1}{2})^{1/2} n(n+1)(x-1)^{1/2} \ln\frac{x+1}{x-1}[1+\mathrm{O}(x-1)]. \tag{A13}$$

Using equation 3.6(5) of Erdélyi *et al* (1953) we obtain from (A4)

$$2^{-1/2}(x+1)^{1/2}(x-1)^{-3/2} = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})^{1/2} Q_n^1(x), \tag{A14a}$$

$$(x+1)(x-1)^{-2} = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})^{1/2} Q_n^2(x). \tag{A14b}$$

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